Modeling Day-to-day Trip Choice Evolution under Network Disruption

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Abstract In this paper we propose a “prediction-correction” framework to model traveler’s perception evolution under network disruption. Distinctive from existing models, the proposed framework assumes drivers make their trip choices according to their predictions on future traffic, since previous daily experiences become invalid when network is disrupted. Drivers predict travel costs after network disruption, and then correct their predictions by comparing with their actual experienced travel costs. Traveler’s prediction is formulated as an individual dynamic process, such that qualitative network conditions could be quantified in the model. We demonstrate the proposed traffic evolution model using the data collected from the I-35W Bridge collapse in Minneapolis, Minnesota, and compare it with a day-to-day traffic assignment model without prediction behaviors. To the best of our knowledge, this is the first time that day-to-day traffic evolution models have been applied to study realistic network disruption scenarios.

1. Introduction

The collapse of the Interstate 35W highway bridge over the Mississippi river in Minneapolis, Minnesota, a major artery in the Twin Cities highway network, constitutes a significant disruption to trip-making patterns in the Twin Cities. While catastrophic to those directly affected by the collapse in terms of fatalities, injuries, and loss of personal property, it is also a rare research opportunity to examine trip-maker learning processes and traffic evolution towards a potential equilibrium or stable state. This introduces a paradigm shift from viewing transportation systems as occupying an equilibrium state in terms of traveler choices, to viewing transportation systems as ones that dynamically evolve towards such fixed states, i.e., systems that undergo an equilibration process.

1.1 A Review of Day-to-Day Traffic Evolution Modeling Literature

Day-to-day (or inter-periodic) traffic modeling methods are believed to be most appropriate for analyzing traffic equilibration processes. With increasing applications of Intelligent Transport Systems (including traveler information and control systems), these day-to-day models were stimulated by capturing day-to-day traffic fluctuations and focused more on the evolution process itself, rather than the final (static) equilibrium state, which is the focus of traditional (deterministic and stochastic) static traffic assignment models. As mentioned by Watling and Hazelton (2003), the most appealing feature to researchers and practitioners is the great flexibility of day-to-day approaches, which allows a wide range of behavior rules, levels of aggregation, and traffic modes to be synthesized into a uniform framework. This equilibration paradigm helps transportation planning and management in modeling evolution of traffic states and the trajectories of evolution.

Traffic equilibration processes can be established on continuous temporal spaces when time steps are "sufficiently" small. Existing continuous time equilibration models employ differential equations to describe traffic evolution. In this category, Smith (1984), Friesz et al. (1994), and Zhang and Nagurney (1996) proposed three dynamical systems. These three systems adopted the assumption of perfect perception of travel cost and develop deterministic traffic assignment processes over a continuous temporal dimension. Specifically, Smith (1984) assumed that travelers on higher travel cost routes will proportionally switch to those routes with lower travel costs. Friesz et al. (1994) proposed a day-to-day model that captures both the dynamics of route flows and origin-destination demands. Zhang and Nagurney (1996) modeled a projected dynamical system, which adjusts day-to-day route flows with a minimum norm projection operator, rather than proportional changes in Smith's work.

Although continuous time approaches have good mathematical properties in traffic evolution, Watling and Hazelton (2003) summarized two major limitations suffered by continuous day-to-day approaches: (1) continuous-time trip adjustment is not plausible in reality; (2) homogeneous population assumptions in these approaches require additional dispersion modules. Therefore, discrete versions of day-to-day traffic equilibration models are more appealing for practical applications, because of the realism of discrete-time adjustments.

In discrete time equilibration systems, traveler route behavior is assumed to be repeated iteration by iteration, in accordance to the daily changes of traffic flows in practice. Specifically, Friesz et al. (1994) employed a projection-type discretization algorithm, given by Bertsekas and Gafni (1982), to approximate the continuous traffic trajectories in the dynamical system developed therein. Nagurney and Zhang (1997) specified their continuous model in a discrete temporal space with fixed demand and applied Euler's method to solve the projected dynamical system. Recently, Yang and Liu (2007a, b) quantified the proportion in Smith's dynamical system. Their model realizes the traffic evolution by assuming a two-stage behavior rule which is a function of traveler experience in the previous day.

Due to the randomness existing in transportation demand, traveler choice and path cost, stochasticity has been introduced into day-to-day approaches when perturbation is under consideration. The majority of existing stochastic assignment models followed Markov processes, as those models proposed in Cascetta, (1989) and Hazelton and Watling (2004), where the computation of transition matrices depends only on the previous traffic state. The transient probability matrix could be specified and leads to approximations of system mean dynamics, as shown by Daganzo and Sheffi (1977), Davis and Nihan (1993) and Yang and Liu (2007a, b). Other stochastic approaches, e.g., Horowitz (1984), Canterella and Cascetta (1995), Watling (1999), adopted the assumption of memory length, assuming that route choice probabilities depend on weighted averages of experienced travel times. To solve these models, Davis and Nihan (1993) provided a particular Gaussian multi-variant autoregressive process and Hazelton et al. (1996) proposed a Markov Chain Monte Carlo method.

In addition to modeling traveler choice behavior, some research focused on traveler learning mechanisms. Traveler learning processes typically emerge when travelers are faced with a new environment, where travelers complete trips, accumulate experience, and adjust choices to minimize disutilities.

Horowitz (1984) studied how learning mechanisms affect traffic state evolution trajectories. Three different learning mechanisms were discussed to interpret three route cost perception processes: (1) weighted averages of measured travel costs, (2) weighted averages of perceived route costs, and (3) unchanged perceived costs unless actual usages. For these three learning mechanisms, the numerical results on a two-link network depicted considerable differences, even though they were applied on the same deterministic route choice model. Majority of previous researches on day-to-day modeling followed one of the learning mechanisms proposed by Horowitz (1984). The commonly accepted and simplest specification of learning mechanisms is that travelers' route choices only depend on their experienced (measured) travel costs on the previous day, e.g., Nagurney and Zhang (1997), Watling (2004), and Yang and Liu (2007a, b). While other studies, e.g., Chang and Mahmassani (1988), Cascetta (1989), Davis and Nihan (1993), adopted the more general weighted averages learning process. In contrast to Horowitz' learning mechanisms, Chen and Mahmassani (2004) employed the Bayesian theory in travel time perception and learning mechanisms, where learning processes can be triggered or terminated under given conditions.

With regard to the importance of information impacts on traveler perception, many day-to-day assignment approaches also considered information effects by combining pre-trip choice and en-route choice into a uniform framework. Such models are generally recognized as doubly-dynamic models. For example, Chang and Mahmassani (1988) applied within-day traffic simulation to provide travel time predictions which result in traveler departure and route adjustments from day to day; Mahmassani (1990) summarized a series of experiments to investigate traveler route and departure time choice behavior using information systems; Cascetta and Cantarella (1991) adopted the discrete choice model to formulate traveler within day choice; Polak and Hazelton (1998) proposed a learning mechanism explicitly considering the en-route information impacts on traveler choices; a similar mechanism also appeared in Yu (2002), where traveler's within-day perceptions of actual travel cost and real-time information affect their choices the next day; Srinivasan and Guo (2004) considered a disaggregated model covering traveler day-to-day departure time choice behavior. Many meso-scopic traffic simulation frameworks, e.g., Hu and Mahmassani (1997) and Jha et al. (1998), consider both route choice and departure time choice, along with information update processes.

Severe network disruption, such as the collapse of the Interstate 35W highway bridge, significantly changes network topology, which should be treated as common information accessible by all travelers. Traveler choice behavior, particular their learning process in days after network disruption, have special characteristics. Because this learning mechanism plays a critical role in the traffic equilibration process, we dedicate this study to traveler learning mechanisms after network disruption.

1.2 Problem Statement and Research Motivation

In the literature, few studies were dedicated to study traffic equilibration after network disruption. As noted by Mahmassani (1990), the major difficulty is that of obtaining observational evidence of real-world traveler choice behaviors. Most day-to-day dynamic traffic models (e.g., Mahmassani et al., 1986; Jotisankasa and Polak, 2005; Kim et al., 2006) rely on experimental approaches and simulations rather than field data, making it necessary to verify their capabilities in the real world. Other studies (e.g., Hunt et al., 2002) only focused on empirical observations of traffic fluctuation under network disruption. We will fill this gap by comparing day-to-day traffic assignment results with field data, in order to specify, estimate, and validate models of day-to-day traffic dynamics. Our aim is to develop a day-to-day traffic equilibration model that can capture aggregate traveler behavioral choices responding to disruptions.

A small-scale survey of travelers who may have used the I-35W Bridge on a daily basis or who may have used routes that were affected by the rerouted traffic was conducted. The specific targets were drivers exiting parking ramps at multiple locations in Minneapolis, and persons waiting at bus stops near those ramps. The survey questions were designed to reveal subjects' day-to-day travel choices for morning commutes, and the factors underlying their choices, before and after the bridge collapse. From the survey results, traveler behavior choices changed significantly immediately after the bridge collapsed. On the day after the bridge collapse, about 46 percent respondents departed earlier than before and 42 percent of respondents changed their routes. However, the detector data of that day showed that the total number trips on freeways actually dropped. From this point of view, traveler behavior did not rely much on the previous days' experiences, but more on "expectations" of the future. Although some studies (e.g., Hunt et al., 2002) provided some preliminary results of traveler responses to network disruption, few of them focused on modeling traveler's trip choice evolution under such disruption cases.

Ben-Akiva et al. (1991) explored the information impacts to traveler choice behavior. Three potential effects, discussed by the authors, were represented as: *oversaturation*, *overreaction*, and *concentration*. In summary, *oversaturation* is when travelers cannot efficiently filter useful information from the large amount of information they are provided with; *overreaction* is when too many travelers respond to current information on traffic conditions; *concentration* describes the case that uniform predictive information can induce consistent selection of the best alternatives. To capture the importance of pre-trip information, Ben-Akiva et al. (1991) proposed a framework, whose learning and forecasting filter contains three basic components of traveler learning: information acquisition, processing capacity, and computational ability. Their work provided a conceptual framework and adopted a traditional weighted average approach as information filters in a stochastic assignment process, to describe how those three effects appear in reality.

Although the previous work in Ben-Akiva et al. (1991) showed interesting results, some major problems have not yet been answered. First, can the simple weighted average approach reasonably represent traveler prediction behavior? Second, how traveler's perception filter and learning process, under network disruption, can be combined analytically in a plausible manner? In other words, what is a reasonable process capturing the learning mechanism inherent in travelers' minds under network disruption scenarios? Although the framework of learning and forecasting filters in Cantarella and Cascetta (1995) has contained the general representations applied in day-to-day modeling, it always assumes that traveler predictions (or estimations) are based on information and experiences in previous day(s). In reality, such predictions should not be dependent only on experiences in previous days. Expectation of traffic pattern changes may have considerable impacts on traveler choices on the next day, especially under severe events. For those scenarios with significant change to the network topology, e.g., major link break (bridge collapse) or new major link opening, the information should be treated as common knowledge to all network users. *Overreaction* and *concentration* may happen. Moreover, since the topology change is significant, travelers' previous experiences may cease to be accurate. Travelers are provided with a large amount of new information, which results in *oversaturation*. Thus, it is plausible to assume that travelers restart a new learning process which relies upon information they received, and/or their own estimation. Drivers make choices using their own predictions of traffic conditions based on available qualitative information. Thus prediction of future traffic pattern affects traveler choices more than their experience in the following days before traffic patterns stabilize, or match traveler perceptions. Further, since effects of *concentration* may be predictable to travelers, *overreaction* may not be realized in reality.

This research will be dedicated to a new day-to-day traffic evolution framework, specifically modeling traveler learning mechanisms under network disruption, using the I-35W bridge collapse as a natural example. We endeavor to provide the conceptual, mathematical and numerical details encompassing this topic. Traveler's learning mechanism after network disruption is the essence of our proposed equilibration process. This process represents typical characteristics of traveler learning. Furthermore, the proposed learning mechanism will exclude details which are irrelevant to the current research purpose, before we go further and capture more realism. Thus, rather than providing a model being more realistic, we focus our study on the simplest process that captures drivers learning characteristics after network disruption.

To the best of the authors' knowledge, many existing day-to-day traffic evolution models, while well-grounded in theory, have not been validated in real-world scenarios. In this context, two challenges arise: (i) dealing with large real-world networks, their sizes make this type of analysis computationally burdensome; and (ii) capturing a real-world evolutionary dynamic goes beyond the convenient assumptions of weighted averages of memories as representative of traveler perceptions. It is, therefore, the main goal of this work, to gain insight into the evolution of trip choices given a real-world example. To examine trip-maker responses to the I-35W bridge collapse, three primary data sources are synthesized: (i) the results of a small scale survey, undertaken by the authors of this paper; (ii) loop detector data from the Twin Cities freeway system; and (iii) arterial tube counts collected by the City of Minneapolis .

2. Trip Choice Evolution under Network Disruption

Cantarella and Cascetta (1995) summarized both deterministic and stochastic day-to-day assignment models into a uniform framework, in which modeling day-to-day traffic equilibration processes includes three major components: a supply model, a learning and forecasting filter, and traveler choice behavior. We follow this fundamental framework to model the traffic evolution under network disruption.

2.1 Supply Model

Assume that traveler trip choice evolution is considered on a given directed network, denoted as $G\left(N,L\right)$ with node set N and link set L. A subset of nodes is considered as centroids (or zones), which generate and attract trips, and thus, represented as origins and destinations. A set of paths, denoted as P^w, connect one origin-destination pair with index w. The total number of trips (or demand) for OD pair w at day t is represented by d_w^t. The demand vector for all OD pairs is denoted by $\mathbf{d}^t$. The path flow on path $p \in P^w$ at day t is denoted by f_p^t. All path flows at day t can be represented by one simple vector $\mathbf{f}^t$. Let x_a^t represent link flow on link $a \in L$ at day t. The vector of link flows is denoted by $\mathbf{x}^t$. Let $\mathbf{A} = \left(\delta_{ap}\right)$ represent the arc-path incidence matrix, then $\mathbf{x}^t = \mathbf{A}\mathbf{f}^t$. Let $\mathbf{\Phi} = \left(\phi_{ip}\right)$ represent the OD-path incidence matrix, then $\mathbf{d}^t = \mathbf{\Phi}\mathbf{f}^t$. Let $\mathbf{c}^t$ denote the link cost vector at day t, with individual link cost c_a^t. If we assume that the link costs are functions of the whole link flow vector, then $\mathbf{c}^t = C\left(\mathbf{x}^t\right)$, where $C\left(\bullet\right)$ can be a given deterministic function, for simplicity. Thus, if we let $\mathbf{F}^t$ represent the path costs vector, with individual path cost F_p^t, then $\mathbf{F}^t = \mathbf{A}^{'}\mathbf{c}^t = \mathbf{A}^{'}C\left(\mathbf{A}\mathbf{f}^t\right)$, where $\mathbf{A}^{'}$ is the transpose of $\mathbf{A}$.

2.2 Modeling the Learning Mechanism

The learning mechanism describes travelers' trip cost perception updating process. Since drivers do not know the actual trip costs before trip completion, their choice behavior is generally assumed to constitute their own pre-trip perceptions. Let Y_p^t represent the (individual expected or average) pre-trip cost prediction on path p at day t, and $\mathbf{Y}^t$ be the pre-trip path cost prediction vector. Thus, existing learning mechanisms generally assume that the pre-trip perceptions $\mathbf{Y}^t$ are based on the previous day's actually experienced costs $\mathbf{F}^{t-1}$ and perceptions $\mathbf{Y}^{t-1}$. Such general learning mechanism is valid for most day-to-day traffic evolution processes, under situations without network topology changes (or when such change is minor). However, this is not valid under severe network disruption. In the latter case, such information significantly affects traveler trip choices. If we let G^t represent such network topology information provided to travelers at day t before trip choices are made, then traveler pre-trip perceptions $\mathbf{Y}^t$ actually are functions of their previous experiences $\mathbf{F}^{t-1}$ and perceptions $\mathbf{Y}^{t-1}$, and currently available information G^t. Thus we may represent the learning mechanism as:

$$\mathbf{Y}^t = Y\left(\mathbf{F}^{t-1}, \mathbf{Y}^{t-1}; G^t, t\right). \qquad (1)$$

Note here $\mathbf{Y}^{t-1}$ can be an implicit function of traffic patterns in finite previous days, so does $\mathbf{Y}^t$. And the previous experienced route costs $\mathbf{F}^{t-1}$ are implicit functions of path flows on the previous day $\mathbf{f}^{t-1}$. However, several difficulties will arise for considering the learning mechanism as presented by (1). First of all, available information G^t is generally qualitative, rather than quantitative in reality. Quantifying such information is required to establish a mathematical traffic evolution model. We will discuss this issue later. Second, a good representation of the learning mechanism needs to be addressed to describe how travelers typically processing available information, under network disruption cases.

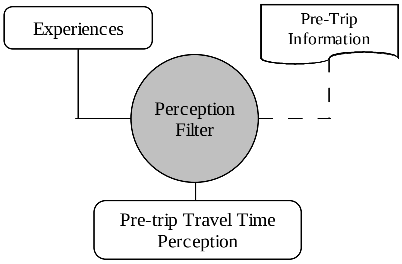

Fig. 1. Regular Traveler Perception Process

Before we describe the travelers' perception behaviors under network disruption, let us first review the regular traveler perception process typically appearing in existing day-to-day traffic evolution models. As shown by Fig 1, traveler perceptions are assumed to be based on previous experiences only. The experiences component in the figure could be either from finite days or from the previous day only. Pre-trip information was also under consideration in a few day-to-day evolution processes. Such pre-trip information was generally assumed to be provided by traffic management units and be received by travelers through subscribed services or variable message systems. In existing day-to-day traffic evolution models, traveler perception was generally modeled as a weighted average of the travelers' memories and the given pre-trip information, where the pre-trip information was assumed having been quantified. Such a framework is valid for regular day-to-day traffic evolution; however it does neglect some characteristics appearing only under network disruption cases.

One special characteristic of trip choice behavior under network disruption is that travelers behave based more on traffic predictions. Especially under unexpected events, as I-35W bridge collapse, significant changes of network topology are generally unpredictable. Thus we revise the existing learning mechanism as a "prediction and correction" process.

Evidence of the existence of prediction and correction behaviors is from the small survey, deployed for studying travelers' behavior changes after I-35 Bridge collapse. For the respondents claimed affected by bridge collapse (77 out of 141 respondents), 75 percent of them (58 out of 77 respondents) changed their departure time in the day right after bridge collapse (Aug. 2, 2007). A believable reason is that travelers *predicted* they may encounter congestions in their commute trips because of the bridge collapse. Without prediction, most travelers would maintain their regular departure time as usual. Corresponding data from loop detectors (station #785) on I-94 westbound around Franklin Ave (which is a section of the designated detour after I-35W bridge collapse) supports this observation as well. As shown in Fig3(b), the AM peak traffic switched to earlier time intervals, such that the traffic volumes at 6:00-7:00 am kept increasing in the days after bridge collapse, although the traffic volumes at 7:00-9:00 am were actually lower than before. However, one week later, 17 percent of them (10 out of 58 responders who had changed their departure times) changed their departure times back to the original departure times before the bridge collapse. It is reasonable to assume that the travelers' actual experiences *corrected* their biased predictions of traffic congestions, such that they switched their departure times to be original ones. This can also be demonstrated by the traffic volumes drop at 6:00-7:00 am in the week Aug 13 to Aug 17, 2007.

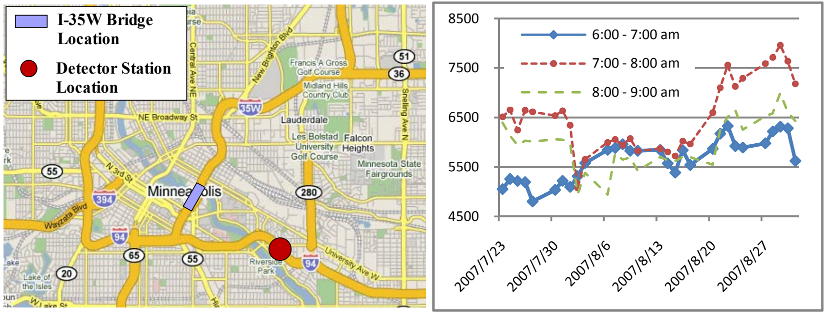

Fig. 3 (a). Location of Detector Station #785 **(b).** Hourly Volumes through Station #785

Based on the observations, it is too optimistic to expect travelers to have enough experience with responding to unanticipated network disruption. Moreover, management units may not provide effective predictive information, due to the unpredictable nature of such events. Furthermore, if such predictive information can be broadcast to every traveler, overreaction and concentration may happen. These forces will drive travelers to apply their own predictions, rather than adopting given information.

The other special characteristic of travelers' trip choice behaviors under network disruption cases could be time-dependency. First, because of the change in network topology, travelers need time to get used to new the environment. In addition, strikes from network disruption can cause some travelers to take time to regulate traffic perception processes. Finally, traveler learning processes need time to stabilize. The prediction behaviors may vanish after travelers accumulate enough experience. Due to such time-dependency properties, the traveler's perception process under network disruption cases should be considered as an adaptive prediction correction procedure. Travelers may have their initial predictive behaviors in the next day right after a significant event. They may also correct (or even abandon) these predictions on the following days after they accumulate experience, and their perception behaviors finally stabilize.

To summarize these two special characteristics of traveler perception processes under network disruption, we depict the traveler learning mechanism using the following flowchart:

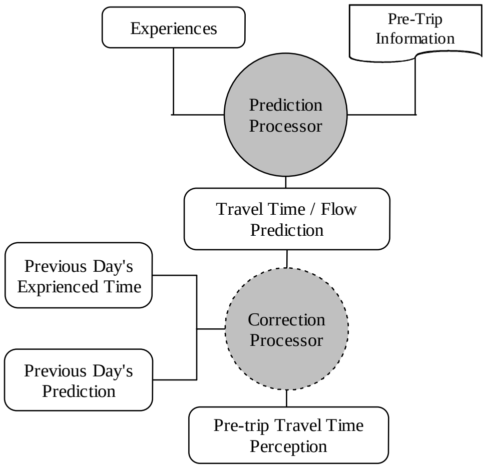

Fig. 2. Traveler Pre-trip Perception Process after Network Disruption

Under this framework, traveler pre-trip travel time perception has two major components: *Prediction* and *Correction*. A traveler estimates a predicted flow pattern based on her/his own experience and available pre-trip information. The solid line used for pre-trip information represents that this is raw information processing by travelers themselves, rather than quantified by traffic management units. In this stage, traveler predictions are calculated by a virtual "*prediction processor*". Traveler experience at the beginning of network disruption may be limited. Travelers may either trim too much useful information or rely too much on irrelevant experience. Thus, their predictions may be biased in the very beginning. After travelers complete their trips in the disrupted network, they receive new experience from the new network. Then, on the following days, in addition to predicting the network flow patterns, travelers include a correction component in the travel time perception process. Comparison between predictions and actual experience now plays its role in correcting biased predictions after travelers gain new experience. These comparisons help travelers refine the travel time predictions achieved from the "prediction processor", allowing travelers to receive better travel time perceptions for the future. Such a correction strategy also appeared in the research of departure time choice dynamics (Mahamassani and Chang, 1986; Chang and Mahamassani, 1988). In contrast, we introduce this correction process in an aggregated level as one component, "*correction processor*", to the whole travel time perception process. Along with the passing time, travelers accumulate their perceptions by their potential learning processes. Both prediction and correction components will gradually vanish in the learning process, when travelers face no supply-side changes (in network topology, controls, or information systems). The dashed border of the correction processor represents the vanishing in this framework when traveler predictions are consistent with their experiences in reality (or, traffic predictions settle to what travelers actually experience), and traveler travel time perception process transforms into a regular one (see Fig 1).

2.3 Traveler's Choice Adjustment

The final component in the traffic evolution process is the traveler's choice model, which is generally presented as a master equation. Cantarella and Cascetta (1995) provided a general form of the master equation as:

$$E\left[\mathbf{f}^t \middle| \mathbf{f}^{t-1}\right] = \mathbf{S}^t \mathbf{f}^{t-1}, \tag{2}$$

where $\mathbf{S}^t = S\left(\mathbf{Y}^t, \mathbf{F}^{t-1}, \mathbf{Y}^{t-1}\right)$ represents the path transition matrix, depending on currently predicted path costs, and previous day actual and predicted path costs. The majority of Markovian-process-based day-to-day equilibration models are specific representations of the master equation. For instance, based on Smith's dynamical system, Yang and Liu (2007b) specified switching choice matrix $\mathbf{W}^t$ and route choice matrix $\mathbf{R}^t$ and proposed a typical two-stage day-to-day adjustment process, where $\mathbf{S}^t = \mathbf{R}^t \mathbf{W}^t + (\mathbf{1} - \mathbf{W}^t)$. Based on different assumptions, traveler's other choice behaviors (e.g., departure time choice and mode choice) can be incorporated into the transition matrix, and result in different traffic evolution models. Since traveler choice behavior model is out of the scope of this research, we will simply adopt existing models herein.

3. Traffic Evolution Process Realization

Since the focus of this study is traveler perception, we will concentrate the mathematical formulation on modeling traveler perception dynamics, following the framework proposed in last section. And traveler trip choice adjustment can easily be adopted by inserting the travelers' pre-trip perceptions $\mathbf{Y}^t$ into existing trip choice dynamic models.

3.1 Mathematical Formulation

We here model traveler perception as a dynamic process based on experience and available information, following a "sociodynamical" approach. Personal behaviors have been modeled as sociodynamics in sociology, economics and regional science. Interested readers are referred to a monograph of modeling sociodynamics by Weidlich (2000). In transportation science, traveler cognitive expectations may be applied as individual dynamics. Referred to by Ben-Akiva et al. (1991), Rasmussen (1986) remarked that "effective control of this information pickup is only possible if the individual has available an *internal dynamic representation* of the state of affairs". Following this idea, it is reasonable to introduce an individual level of dynamics for modeling travelers' pre-trip predictions, in order to characterize travelers' behaviors after severe events.

To concentrate our study on modeling travelers' perception dynamics, we only consider route choice in traveler day-to-day choice behavior. Departure time choice could be easily included in the model as well, based on a given traveler learning mechanism and departure time choice model.

Assume that travelers' route choice adjustment process satisfies a dynamical system defined as:

$$\frac{d\mathbf{f}}{dt} = \mathbf{g}\left(\mathbf{Y}\right). \tag{3}$$

This dynamical system defines that traveler route choice behaviors are a set of nonlinear functions of traveler perceptions of the route costs. Due to (1), the perception vector $\mathbf{Y}$ essentially is a function of route flows $\mathbf{f}$, the network topology G, and time t. Thus, nonlinear functions $\mathbf{g}\left(\mathbf{Y}\right)$ could be represented by $\mathbf{g}\left(\mathbf{f};G;t\right)$, where the network topology parameter G may change with time. If network topology is assumed to be constant, denoted by G_0, and a route flow pattern $\mathbf{f}_0$ is a known stable solution of (3), under the given network topology G_0, then $\mathbf{g}\left(\mathbf{f}_0;G_0;t\right) \equiv 0$. Now, because of network disruption, the given network topology shifted from G_0 to a certain $\hat{G}$, and the state variables after the network change, $\mathbf{f}\left(t\right)$, will accordingly have a new form:

$$\mathbf{f}\left(t\right) = \mathbf{f}_0 + \mathbf{w}\left(t\right), \tag{4}$$

where the new system variables $\mathbf{w}\left(t\right)$ is a deviation from the stationary solution $\mathbf{f}_0$ due to the network topology change $G_0 \Rightarrow \hat{G}$. The whole analysis of traveler choice adjustment change is a combination of the initial given stationary process, and a perturbation process which is the major part of this research.

Following the idea of prediction and correction, we can specify the traveler perception adjustment process in a discrete time version as:

$$\mathbf{Y}^t = \hat{\mathbf{Y}}^t + \alpha_t\left(\mathbf{F}^{t-1} - \mathbf{Y}^{t-1}\right), \tag{5}$$

where $\mathbf{Y}^t$ represents travelers' per-trip perception vector on day t, $\hat{\mathbf{Y}}^t$ represents travelers predictions (based on certain stable potential calculation system which we will discuss later), and $\mathbf{F}^{t-1} - \mathbf{Y}^{t-1}$ provides travelers a correction direction with scale $0 \le \alpha_t \le 1$. Actually, the perception adjustment (5) is a weighted average of prediction and a corrected prediction, as:

$$\mathbf{Y}^t = \left(1-\alpha_t\right)\hat{\mathbf{Y}}^t + \alpha_t\left[\hat{\mathbf{Y}}^t + \left(\mathbf{F}^{t-1} - \mathbf{Y}^{t-1}\right)\right]. \tag{6}$$

Their mathematical meanings will be explored later. Traditional perception filters, where $\hat{\mathbf{Y}}^t = \sum_{i=1}^{k} w_i \mathbf{F}^{t-i}$ and $\alpha_t = 0$, do not allow the network topology change $G^{t-1} \Rightarrow G^t$ to impact traveler's pre-trip route cost perceptions on day t, since all experienced route costs $\mathbf{F}^{t-i}$ were realized before the network change. One possible way to include such information is to specify prediction $\hat{\mathbf{Y}}^t$ as given quantified information to all travelers. Even so, quantification of such information is also interesting to traffic management units. Thus, the prediction $\hat{\mathbf{Y}}^t$ should include pre-trip information (network topology or weather etc.) that may impact travelers' perceptions.

Now, we further discuss the representation of the route cost prediction $\hat{\mathbf{Y}}^t$ in the adjustment process (5). Using different assumptions, the route cost prediction vector has different realizations, as does the traffic evolution model. One straightforward assumption is that all traveler route choice behavior rules (responding to any network change) will be common knowledge. Here, common knowledge means that travelers know exactly what other traveler choice strategies are. This assumption also implies that travelers are rational. They have the potential to choose the best responses to any network changes. Based on this assumption, travelers draw their route cost predictions based on knowledge of other traveler route choice strategies, i.e., dynamical system (3). Thus, in a discrete deterministic version, the traveler route cost prediction vector has one representation as:

$$\hat{\mathbf{Y}}^t = \hat{\mathbf{F}}^t = \mathbf{A}'\mathbf{c}^t = \mathbf{A}'C\left(\mathbf{A}\hat{\mathbf{f}}^t\right), \tag{7}$$

where the estimated path flow vector $\hat{\mathbf{f}}^t$ is derived from the dynamical system (3), due to the assumption that all travelers can access it. In details, the calculation of $\hat{\mathbf{f}}^t = \mathbf{f}^{t-1} + \mathbf{g}\left(G^t; \mathbf{f}^{t-1}; t\right)$. Specifically, we can adopt the master equation (2), and let $\mathbf{S}^t = \mathbf{R}^t\mathbf{W}^t + (\mathbf{1} - \mathbf{W}^t)$ represent the two-stage day-to-day adjustment process. Consequently, the function $\mathbf{g}\left(G^t; \mathbf{f}^{t-1}; t\right) = \mathbf{R}^t\mathbf{W}^t\mathbf{f}^{t-1} - \mathbf{W}^t\mathbf{f}^{t-1}$ is nonlinear since $\mathbf{R}^t$ and $\mathbf{W}^t$ are generally assumed to be nonlinear functions of $\mathbf{f}^{t-1}$ and G^t. Then, the traveler prediction vector has the following representation:

$$\hat{\mathbf{Y}}^t = \mathbf{A}'C\left[\mathbf{A}\left(\mathbf{R}^t\mathbf{W}^t + (\mathbf{1} - \mathbf{W}^t)\right)\mathbf{f}^{t-1}\right]. \tag{8}$$

More specifically, Yang and Liu (2007b) proposed one representation of discrete deterministic dynamical system (3), and provided one specification of the switching rate matrix $\mathbf{W}^t$ and route choice matrix $\mathbf{R}^t$. For completeness, we provide the computation of these matrices in the following.

By applying this two-stage day-to-day adjustment process and proportional-switch adjustment process by Smith (1984), the dynamical system (3) for traveler's perception is now represented by:

$$\frac{df_p^w}{dt} = \sum_{q\in P^w} f_q^w \left[F_q^w - F_p^w\right]_+ - f_p^w \sum_{q\in P^w} \left[F_p^w - F_q^w\right]_+ . \tag{9}$$

Here, the dynamical system (9) provides a predicted drivers' route choice behavior for the following day, under the consideration of given network change $G_0 \Rightarrow \hat{G}$.

Yang and Liu (2007b) proposed an approximation of route switching choice probability and route choice probability based on large population number. In detail, the switching rate matrix $\mathbf{W}^t$ is specified by a diagonal matrix, as:

$$\mathbf{W}^t = diag\left(\ldots, \frac{1}{T_w} \sum_{q\in P^w} \left[F_{p,t-1}^w - F_{q,t-1}^w\right]_+ , \ldots \right), \tag{10}$$

where P^w represents the route set that belongs to OD pair w, q and p are route indices, and $\left[\bullet\right]_+$ is a projection operator defined as: $\left[x\right]_+ = \max\left\{0, x\right\}$. In the formulation, $T_w = \sum_{q\in P^w} \sum_{q\in P^w} \left[F_{q,t-1}^w - F_{p,t-1}^w\right]_+ + M$ specifies the switching rates, with a constant reluctance parameter $M > 0$, i.e., more travelers prefer maintaining previous choices when a larger M appears. The route choice matrix $\mathbf{R}^t$ is a $\left(k \times k\right)$ block diagonal matrix, where k is the number of OD pairs. Let $\mathbf{R}_w^t$ represents a $\left(m_w \times m_w\right)$ block route choice matrix for OD pair w, where m_w is the number of paths for OD pairs w. And if we let $r_{p,q}$ represents the element in row p and column q, then

$$r_{p,q} = \frac{\left[F_{p,t-1}^w - F_{q,t-1}^w\right]_+}{\sum_{q\in P^w} \left[F_{p,t-1}^w - F_{q,t-1}^w\right]_+}, \tag{11}$$

which interprets the route choice proportional switching from route q to route p. Based on (10) and (11), traveler route cost prediction can be applied as the following discrete version of dynamical system (9):

$$\hat{f}_{p,t}^w = f_{p,t-1}^w + \frac{1}{T_w}\left\{ \sum_{q\in P^w} f_{q,t-1}^w \left[F_{q,t-1}^w - F_{p,t-1}^w\right]_+ - f_{p,t-1}^w \sum_{q\in P^w} \left[F_{p,t-1}^w - F_{q,t-1}^w\right]_+ \right\}.$$

The discrete version of (9) provides one realization of travelers' route flow predictions $\hat{\mathbf{f}}^t$. Note that the experienced route costs $F_{q,t-1}^w$ and the feasible route sets

P^w are updated at this level, based on the pre-trip network information G^t. If traveler's choice dynamics (3) is formulated as PAP, then the route path flow pattern at day t is:

$$f^w_{p,t} = f^w_{p,t-1} + \frac{1}{T_w}\left\{ \sum_{q\in P^w} f^w_{q,t-1}\left[\hat{F}^w_{q,t} - \hat{F}^w_{p,t}\right]_+ - f^w_{p,t-1} \sum_{q\in P^w} \left[\hat{F}^w_{p,t} - \hat{F}^w_{q,t}\right]_+ \right\},$$

in which $\hat{F}^w_{q,t}$ represent the predicted route costs, determined by (7).

We summarize one realization of day-to-day route choice adjustment, including the prediction and correction behaviors proposed in the previous section, into the flowchart shown in Fig 4. In this flowchart, the numbers above the arrows represent the mathematical formulation realizing the processes from one component to another. Travelers' route choice behavior dynamics, eqn. (3), may have the same representation of route cost prediction dynamics (9), under the assumption on traveler knowledge acquisition.

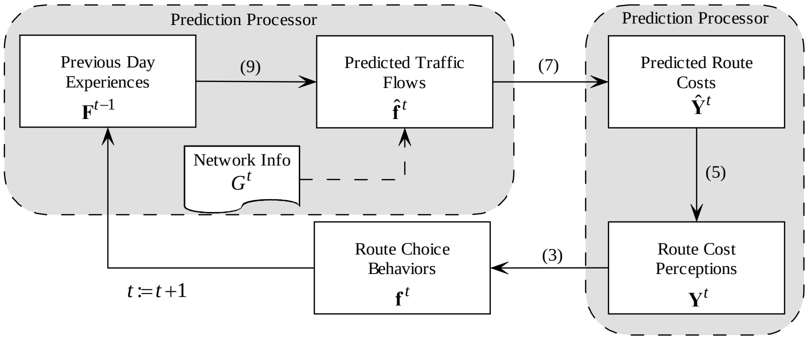

Fig. 4. Implementation Framework of Travelers' Choice Adjustment Process

These prediction and correction components may have other representations differing from what we proposed in this section, by applying different assumptions. First, the dynamical system may have representations, rather than (9), to represent traveler's prediction behavior. For instance, other dynamical systems presented by Friesz et al. (1994), or Zhang and Nagurney (1996). Second, the prediction update process could be different. For example, the prediction behavior may only appear when new information is perceived by travelers. Travelers predict traffic condition once, according to the new network topology change. On the following days, travelers stop predicting but keep correcting their predictions according to their actual experience, until the experienced travel costs match their pre-trip perceptions. Third, predictions may only be made by part of the population. Not all travelers necessarily make predictions. Those travelers, who are not affected by network topology changes, can reasonably be assumed to follow the regular perception process, and change their choices accordingly. Finally, the correction parameter α_t may have different representations, e.g., decreasing monotonically and converging to zero along with time t.

3.2 Properties

The proposed implementation/solution framework of the traveler choice adjustment process could be represented as finding the fixed points of deterministic traveler choice dynamics.

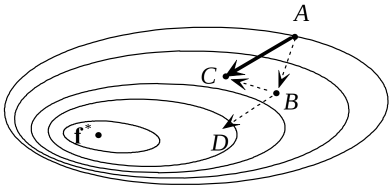

Fig. 5. Approach to Travelers' Choice Fixed Point

Fig 5 above shows a disutility contour figure. A negative gradient direction represents the direction in which travelers can minimize their disutilities. Let $\mathbf{f}^*$ represents the fixed point of travelers' choice dynamics after a certain network disruption. Assume that point A is the stationary fixed point before network disruption, and now become an initial point of the travelers' choice behaviors. When travelers have experiences based on previous day traffic, shown by point A, traditional models assume that travelers behave along with direction AB, which is the negative gradient at point A (that is determined by the dynamical system (3) at point A). However, in our proposed model, travelers are able to predict traffic flow pattern at point B in the following day, and thus able to behave along with another direction of AC, which directs the same as the negative gradient BD (decided by dynamical system (3) at point B). Furthermore, the vector BC provides correction direction of travelers' predictions in the following day.

For the proposed implementation framework, the stability is defined as in previous research (e.g., Smith, 1984) based on mild assumption of the dynamical system (3) and its representation (11). In detail, when the link flow cost functions $C(\bullet)$ are continuously differentiable and monotone, then traveler route choice dynamic process (3) is stable. One notable point is that the traveler perception process proposed herein uses the discrete time version. Different discrete mathematical formulations of traveler perceptions are specific representations of the moving directions of traffic flow patterns. When the time step lengths converge to zero (continuous time), traffic flow patterns always move along the gradient direction, as shown by AB in Fig 5. In other words, direction AC converges to AB in continuous time.

When the prediction process (9) has the same representation as (3), the proposed traveler's learning process is *homogeneous*, as defined by Cantarella and Cascetta (1995). That is:

$$\mathbf{F}^{t-1} \neq \mathbf{Y}^{t-1} \Rightarrow \mathbf{Y}^{t} \neq \mathbf{Y}^{t-1}, \tag{12}$$

This property addresses that traveler keeps adjusting perception if that is not confirmed by experience. Thus, when traveler's route cost perceptions approach actual experienced route costs, prediction and correction gradually vanish, since there are no differences between perceptions and actual experiences.

4. Numerical Example

To demonstrate performances of the proposed leaning mechanism under network disruption, we apply our traffic evolution framework to study the traffic fluctuation after I-35W bridge collapse. Before collapse, more than 140,000 vehicles crossed the I-35W bridge daily. As a major entry of city Minneapolis, this bridge carried nearly 25% of the traffic accessing the downtown area. It was also a primary route to the University of Minnesota, whose northwest campus is located only five blocks away.

The network used is the Twin-Cities Metropolitan Council 2010 planning network. This network contains 25,598 links, and 9,895 nodes, of which 1632 are traffic analysis zones generating and absorbing trips. A snapshot of the testing network is given in Fig 6 (a). And bridge collapse location with respect to the Twin Cities beltway is shown by Fig 6 (b).

Fig. 6 (a). Twin Cities Metropolitan Network **(b).** Bridge Collapse Location

The studying period is from July 23, 2007 to August 31, 2007. Only weekdays are considered. A series of network topology changes took place within this period to accommodate re-routed traffic. In detail, the I-35W Bridge collapsed on August 1, 2007, and freeway ramps around the bridge were closed; after the collapse, the Cedar Ave Bridge remained closed for investigation purpose until August 31; MN-280 was approved as a detour route and was converted to a freeway by blocking off all side-street access; further, the ramps connecting MN-280 and I-94 were expanded to two lanes on August 13; finally, one more lane was added on each direction of an I-94 three-mile stretch between MN-280 and I-35W on August 20.

The traffic demand data applied in this test is a morning peak hour trip table from Metropolitan Council 2010 planning demand model. The projected demands in 2010 were scaled down to reflect trip-desires in 2007. The demand multiplier was determined by assuming that freeway system usage is a constant proportion to the total number of trips. We first summed up the traffic volumes entering the freeway system through on-ramps based on loop detector data, and also on a static traffic assignment result which was implemented by gradient projection algorithm on 2010 planning network using the projected demands. The rate of actual freeway usage to the assigned freeway usage provided the demand multiplier basis in 2007.

Further, the multiplier to the demand table changes along with time, in order to capture the daily trip number fluctuations. The multiplier varies according to the change of total number of trips using freeway system at morning peak hours (from 6 to 9 am). Fig 7 exhibits the fluctuation of total number of trips using the freeway system in twin cities area, within the weekdays from July 23 to August 31, 2007. Each number in the figure is a summation of traffic volumes entering the freeway system through on-ramps, within the morning peak hours each day. Let h^t denote the number of trips using the freeway system on day t. Then the demand vector $\mathbf{d}^t$ on day t is determined by $\mathbf{d}^t = \left(h^t / h^{t-1}\right)\mathbf{d}^{t-1}$. This is done to arbitrarily remove the effects of demand fluctuation and to, thus, facilitate focusing our attention on route choice behavior.

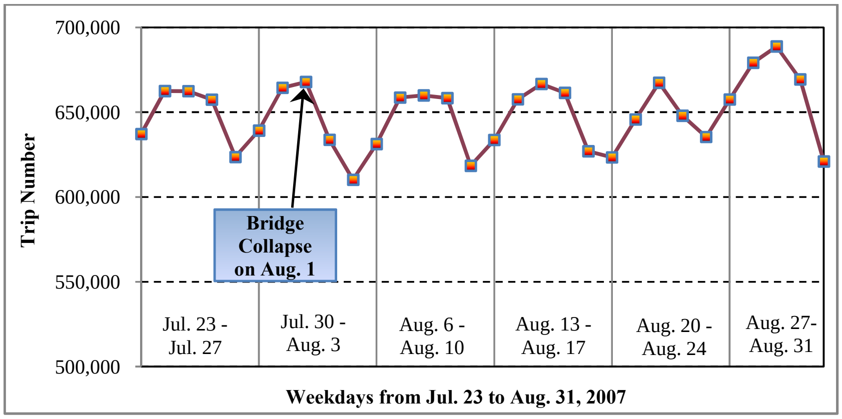

Fig. 7. Total Trips Entering Freeway System at Morning Peak Hours (6-9am)

The day-to-day traffic evolution follows the implementation framework depicted in section 3. The proportional-switching adjustment process (PAP) has been utilized in both the traffic prediction process and the route choice dynamics. The impact parameter α^t is arbitrarily set to be 0.1.

Fig 8 shows the locations where we compare the assignment results with observed traffic volumes. The large dots in the figure are the freeway loop detector locations obtained from the Minnesota Department of Transportation (MnDOT) and the small dots are the arterial tube counts locations obtained from the City of Minneapolis.

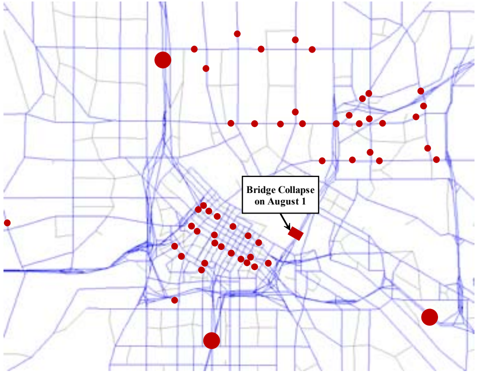

Fig. 8. Traffic Volume Data Collection Locations

The day-to-day traffic assignment results are compared with the observed data collected on freeways and arterials around the city of Minneapolis. The traffic volumes on freeways were collected by loop detectors and the traffic volumes on arterials were collected by tube counts. Since loop detector data were collected on consecutive days, they can clearly illustrate traffic evolution after network disruption.

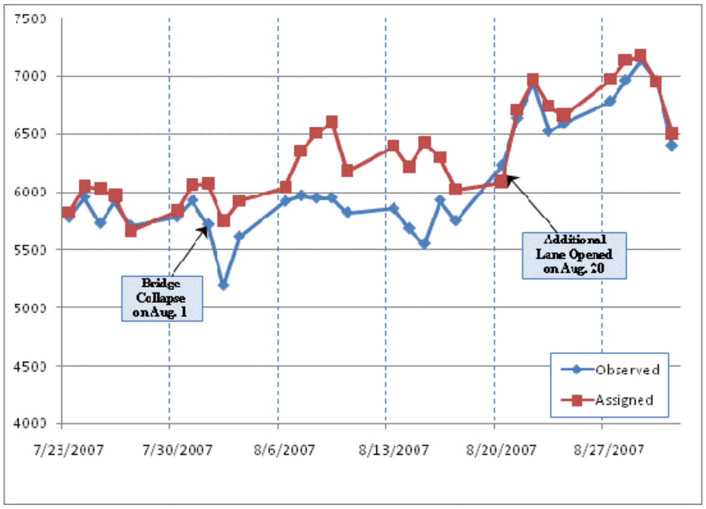

Fig. 9. Average Traffic Volume Evolution on Detector Station #785 (6:00am-8:00am)

Fig 9 shows the day-to-day AM peak traffic evolution on I-94W around Franklin Ave, whose location is identified in Fig 3(a). The observed traffic volumes shown in this figure are average hourly volumes from 6 am to 8 am, in order to capture the departure time changes in reality. Reflected by both detector data and assignment results, many travelers chose to avoid traveling at AM peak hour on August 2nd. They may abandon their trips, change trip departure time, switch trip modes, or alternate to other routes, such that the AM peak traffic on that section was much lower than historical data. Traffic was getting back and settled in a relative stable level in the following two weeks after August 6th, since travelers were getting use to the new environment. In this period, the day-to-day assigned link volumes are higher than reality. The major reason is that the utilized day-to-day assignment model does not include queue impacts. As other planning models, travelers are instantaneously assigned into the network, although traffic volumes already exceed link capacities and queues are starting accumulated. The day-to-day assigned link volumes, in weeks from August 6th to August 17th, do show a fact that I-94 was attracting more travelers than before. However, this attractiveness was not explored by the loop detector data, since queues were piled up on the upstream sections to this location. This argument could be investigated by checking the occupancy data collected on TH280 and I-94W, upstream to this location. Fig 10(a) below shows one detector station (#550) location upstream to the location we compared traffic flow evolution. The average occupancy (from 6 am to 8 am) evolution on this station has been shown by Fig 10(b). As we can see, the occupancy on this location kept high within these two weeks from August 6th to August 17th. It means that a lot of traffic was been queuing up and travelers were not able to complete their trips in AM peak hour. Because of limited loop detector data on TH280, we cannot show the occupancy evolution on TH280, but the occupancy did rise up to 34%, on August 15th. These upstream bottlenecks resulted in a fact that the stable link flows were close to link capacity, as the observed link flows shown in Fig 9. After the new lane opened on August 20th, congestions around this section have been removed, demonstrated by a lower occupancy on upstream freeway section shown by Fig 10(b)(detector station #550). As exhibited by Fig 9, more traffic went through detector station #785 in the AM peak. Traffic attraction of this section, also reflected by the day-to-day assignment results in advance, finally realized.

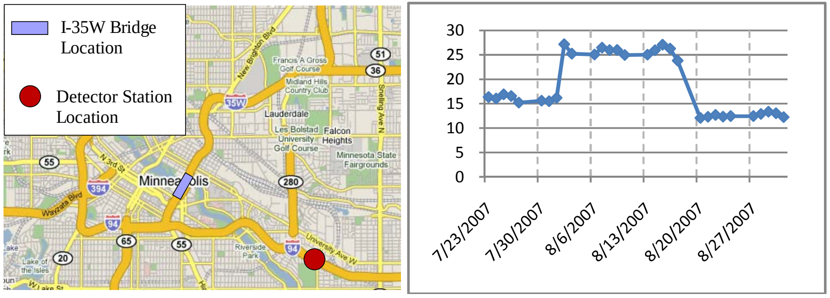

Fig. 10. (a) Location of Detector Station #550 **(b)** Occupancy Evolution on Station #550

Finally, if we compare the numerical results with the realistic data, current day-to-day assignment model performs well, considering the difficulty of implementing on a huge size network and calibrating traffic demand. Further, traveler's other perceptions, e.g., number of stops and scenery feelings were not considered in this model. The scatter diagram of Fig 11 exhibits a comparison between day-to-day assignment results and observed data. The R-Squared value of 0.9168 provides a good index of the fitting results.

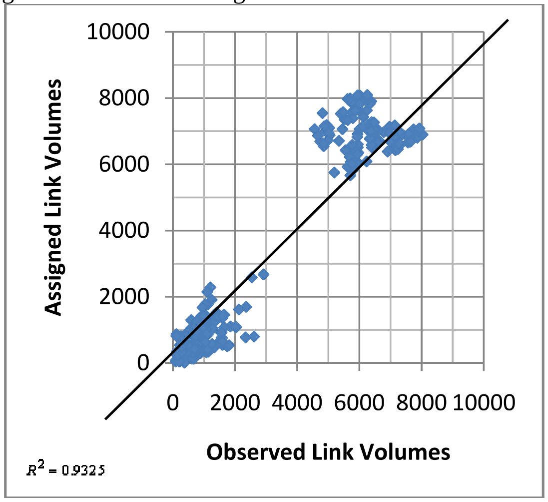

Fig. 11. Comparison between Observations and Assignment Results

5. Conclusions

We proposed a "prediction and correction" process to model traveler's perception dynamics after network disruptions. In contrast to traditional perception filters, the proposed model assumes travelers having the ability to evaluate possible traffic conditions in the days following network topology changes. Traveler perception changes are modeled as an individual dynamical system, allowing traffic patterns to evolve along "better" directions that help travelers to avoid concentration and overreaction. This is the first time day-to-day traffic assignment models are applied to a real-world evolutionary dynamic. The computational example shown in this paper highlights the fact that the learning mechanism can have a profound effect on the evolution of traffic flows.

Some extensions could be considered in future study. First of all, we only focused on a deterministic process. Comparing deterministic with a probabilistic evolution, we must state that the latter is more appropriate for representing traffic dynamics. The probabilistic can allow for unobservable features by drawing from an appropriate distribution, and incomplete knowledge can be introduced to the model. Second, the parameters and traffic demands are arbitrarily determined, without any calibration. Statistical estimation techniques could be applied to for parameter calibration. Third, the perception dynamic is considered at an aggregated level. Studying traveler perception updates can be implemented at a disaggregated level by using simulation, as in Hazelton et al. (1996). Finally, bounded rational behavior adjustment processes could simply be modeled in the proposed framework, by introducing a small negative perception bound into projection operator $[\bullet]_+$ in travelers' prediction dynamics (9).

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